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FZZ algebra

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Abstract: The duality between the Sine-Liouville conformal field theory and the two dimensional black hole is revisited by considering the two possible Sine-Liouville dressings together. We show that this choice is consistent with the structure of correlation functions, and that the OPE of the two dressings yields the black hole deformation operator. As an application of this approach, we investigate the role of higher winding perturbations in the context of $c = 1$ strings, where we argue that they are related to higher-spin discrete states that generalize the 2d black hole operator.

Keywords: Black Holes in String Theory, Conformal Field Models in String Theory.

Contents

1. Introduction

It has long been known that the bosonic string admits a two-dimensional black-hole like background, described as a gauged $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ WZW model [1] and can also be thought of for some values of the parameters as a solution of the lowest order (in α') effective action [2, 3]. Moreover, it was shown that when viewed as a perturbation of the $c = 1$ string theory, the leading term in this solution uniquely extends to a full solution of closed string field theory [4].

Some years ago, Fateev, Zamolodchikov and Zamolodchikov [5] proposed that the gauged $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ CFT has a dual description in terms of a free theory (with a linear dilaton) perturbed by a Sine-Liouville potential. This remarkable relation between two seemingly different models, the so called FZZ duality, has been explored and applied in several ways (see for example [6 – 11]). The duality has an $N = 2$ supersymmetric version [12 – 14], as well as a realization on the boundary of the worldsheet [15, 16].

On the other hand, progress made during the last few years in the study of non-rational conformal field theories (see [17 – 19] for reviews) has shown that both dressings of Liouville-like perturbations in linear dilaton theories appear in the exact solutions [20]. The latter typically have two classical limits, and in each limit one of the two perturbations disappears. This suggests that the classically vanishing operator is a non-perturbative quantum effect generated by the backreaction of the first one.

Therefore it is natural to consider a Sine-Liouville theory where both dressings are taken into account and to ask how the FZZ duality fits in such a setting. In this work we propose an answer to this question which gives a new perspective on the FZZ duality. Our approach is based on the observation that the OPE of the two Sine-Liouville dressings yields the black hole perturbation. Therefore the latter operator closes a sort of algebra which we have dubbed the FZZ algebra. In order to preserve the exact marginality of the perturbations, the black hole operator should then be added to the action. Finally, a perturbative computation will show that the coefficient in front of the second sine-Liouville should be put to zero. In this way, the standard form of the FZZ duality is recovered, with just one Sine-Liouville perturbation along with that of the black hole.

This approach to the FZZ duality suggests in turn a natural generalization of the FZZ algebra in the $c = 1$ non-critical string context. This is motivated by the fact that in this case, the two-dimensional black hole is the first of an infinite family of solutions to the closed string field theory equations, each one corresponding to one of the discrete states of the $c = 1$ string [4]. We find that when Sine-Liouville perturbations with different winding numbers are turned on, all the discrete states of the $c = 1$ can be generated by multiple OPEs. This strongly points to the existence of an infinitely generalized FZZ duality in the $c = 1$ string, which should be further investigated.

The organization of this work is as follows. In section 2 we briefly review the two-dimensional black hole background and the FZZ duality. In section 3 we introduce the FZZ algebra, we show that all the interactions involved are compatible with the parafermionic symmetry of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset and that the second Sine-Liouville dressing is consistent with the correlation functions of the theory. In section 4, we briefly review the $c = 1$ string theory and present our proposal for the enlargement of the FZZ algebra in this model. Section 5 contains the conclusions.

2. Euclidean 2d black hole and FZZ duality

2.1 2d black hole - review

We start by reviewing some basic properties of the two dimensional cigar or black hole solution in noncritical string theory [1 − 3] This will also serve to establish some notation and conventions.

This black hole solution can be written as an exact conformal field theory (all orders in α'), namely an $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ WZW model [1], whose Euclidean version is a σ-model with metric:

$$\begin{aligned} ds^2 &= k\left((1-e^{2Q\phi})\,dt^2 + \frac{1}{1-e^{2Q\phi}}d\phi^2\right) , \\ \Phi - \Phi_0 &= Q\phi, \qquad -\infty < \phi < 0 \ . \end{aligned} \tag{2.1}$$

Here k is the level of the $\mathrm{SL}(2,\mathbb{R})$ WZW model and $Q = \frac{1}{\sqrt{k-2}}$.

By a change of coordinates, this solution can also be written:

$$\begin{aligned} ds^2 &= k\left(dr^2 + \tanh^2 r\, d\theta^2\right) \\ \Phi - \Phi_0 &= -2\log\cosh r, \qquad -\infty < r < 0\,. \end{aligned} \tag{2.2}$$

The geometry of the Euclidean black hole is that of a cigar ending at $r = 0$. Its asymptotic radius as $r \to -\infty$ is

$$R = \sqrt{k}\,. \tag{2.3}$$

The value of the dilaton at the tip, Φ_0, can be identified with the mass of the black hole:

$$M \sim e^{-2\Phi_0}\,. \tag{2.4}$$

The 2d black hole can either be considered by itself as a string background, or adjoined to another "internal" CFT to form the total string background. In the former case, conformal invariance of the worldsheet theory requires:

$$c_{\text{tot}} = \frac{3k}{k-2} - 1 = 26 \quad \Rightarrow \quad k = \tfrac{9}{4} \quad \Rightarrow \quad R = \tfrac{3}{2}\,. \tag{2.5}$$

Therefore in this case one is in the regime of small k, and the spacetime solution is not very reliable. On the other hand, if we add an internal CFT then it is easy to see that k can be arbitrarily large and one expects the spacetime solution to be a reliable guide to the physics. In this and the next few sections we will assume the most general situation, with k arbitrary. Later we will specialise to the case where there is only a black hole and no internal CFT.

An important role will be played by the fact that the 2d black hole background has a parafermionic $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ symmetry. Note that a cosmological Liouville perturbation would spoil this symmetry. Hence we assume there is no cosmological perturbation, which is physically acceptable since the would-be strong coupling region is already cut off by the black hole geometry.

For large negative ϕ, the black hole metric can be written

$$\begin{aligned} ds^2 &= k\Big((1 - e^{2Q\phi})\, dt^2 + (1 + e^{2Q\phi})d\phi^2\Big)\,, \\ &= k\Big(dt^2 + d\phi^2 - (dt^2 - d\phi^2)e^{2Q\phi}\Big)\,. \end{aligned} \tag{2.6}$$

Thus, infinitesimally the black hole is generated by a perturbation

$$\Delta S = (\partial X\bar\partial X - \partial\phi\bar\partial\phi)e^{2Q\phi}\,. \tag{2.7}$$

The second term is a pure gauge in BRST cohomology. Therefore the black hole background is generated by the operator:

$$B = \partial X\bar\partial X e^{2Q\phi}\,. \tag{2.8}$$

It should be kept in mind that this operator only describes the 2d black hole far away from the horizon, in the weak-coupling region $\phi \to -\infty$. However, it unambiguously generates the full solution, in the sense that a CFT perturbed by B will flow to the CFT of the

JHEP01(2007)025

Euclidean 2d black hole [4]. In this process the spacetime gets cut off at the horizon, leading to the well-known property that winding number is violated: a string wrapped around the Euclidean time direction in the asymptotic region can be slipped off at the horizon. Violation of winding number is not, however, evident from inspection of the operator B, which by itself conserves winding number.

2.2 FZZ duality

The FZZ duality [5] states that the Euclidean 2d black hole discussed above is "dual" to the Sine-Liouville perturbation of the linear dilaton theory.

The latter arises by coupling a compact "matter" coordinate X to the Liouville field. Since X is compact, it can be split into $X = X_L + X_R$. Let us normalize the holomorphic fields X_L and $\phi(z)$ as ($\alpha' = 1$)

$$X(z)X(w) \sim \phi(z)\phi(w) \sim -\frac{1}{2}\log(z-w)\,. \tag{2.9}$$

and similarly for the anti-holomorphic fields $X_R, \phi(\bar{z})$. The worldsheet stress tensor is

$$T = -(\partial X)^2 - (\partial\phi)^2 + Q\partial^2\phi, \tag{2.10}$$

with

$$Q = \frac{1}{\sqrt{k-2}}\,, \tag{2.11}$$

and the central charge is

$$c = 2 + 6Q^2\,, \tag{2.12}$$

which is the same as eq. (2.5). The linear dilaton is given by

$$\Phi - \Phi_0 = Q\phi\,, \tag{2.13}$$

so that, with $g_s = e^{\Phi_0}e^{Q\phi}$, the theory is weakly coupled as $\phi \to -\infty$.

The vertex operators of this theory are written:

$$V_{\alpha,\beta} = e^{2i\alpha X}e^{2\beta\phi}\,, \tag{2.14}$$

and have conformal dimension

$$\Delta = \alpha^2 + \beta(Q-\beta)\,. \tag{2.15}$$

The wave function corresponding to these operators is obtained by multiplying by $g_s^{-1} \sim e^{-Q\phi}$. It follows that whenever $\beta < \frac{Q}{2}$ the wave function is non-normalizable, in that it is peaked about the weak-coupling region $\phi \to -\infty$. This is sometimes called the "allowed" dressing. Its insertion creates a local deformation of the worldsheet. For $\beta > \frac{Q}{2}$ the wave function decays at weak coupling and is normalizable, and its insertion creates a non-local deformation.

If the theory is perturbed by an operator that creates a "wall" at strong coupling, the situation is different. Only one linear combination of right-moving and left-moving waves survives. As a consequence, the corresponding Euclidean operator will be a linear combination of normalizable and non-normalizable ones.

Now let us introduce the Sine-Liouville perturbations:

$$\mathcal{T}^{+}_{\pm R} = e^{\pm iR(X_L - X_R)}\, e^{(Q-|Q-\frac{1}{Q}|)\phi}\,, \tag{2.16}$$

where as before, $R = \sqrt{k}$. The subscript labels the "winding momentum" for the matter part of the vertex operator, while the sign in the superscript labels the Liouville dressing. In particular, the above operators both have the "allowed" value of the Liouville dressing, so that the corresponding wave-functions grow at weak coupling and are non-normalizable. These operators carry winding number ± 1 around the Euclidean time direction.

The FZZ duality states that the 2d black hole theory is equivalent to Sine-Liouville. One of our goals in what follows will be to make this notion more precise. However first let us review the existing evidence for this duality. It comes from the knowledge of the exact two- and three-point functions (on the sphere) of the 2d black hole theory. For example, the two-point function, which we will re-obtain below, is:

$$R(j,m,\bar{m}) = \left(\frac{\mu\pi\Gamma(\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}\right)^{1-2j} \frac{\Gamma(2j-1)\Gamma(1+\frac{2j-1}{k-2})}{\Gamma(-2j+1)\Gamma(1-\frac{2j-1}{k-2})} \frac{\Gamma(-j+1+\bar{m})\Gamma(-j+1-m)}{\Gamma(j+\bar{m})\Gamma(j-m)}\,. \tag{2.17}$$

The poles in the first two Γ-functions of the numerator reflect the noncompact nature of the target space [21]. It can be shown that the positions of the poles of the first Γ-function, occurring at

$$j = 0, -\tfrac{1}{2}, -1, -\tfrac{3}{2}, \ldots \tag{2.18}$$

can be obtained using the black hole operator as a screening charge, and the residues at these poles can be computed using free field techniques. Together this determines the above correlator. On the other hand, the poles of the second Γ-function, at

$$1 + \frac{2j-1}{k-2} = 0, -1, -2, \ldots \tag{2.19}$$

can be obtained using the Sine-Liouville operator as the screening charge, and their residues again give the remaining factors in the correlator. The agreement has also been shown to hold for three point functions in [7, 9], for processes conserving and violating winding number.

It is intriguing that this duality works quite similarly to channel duality in critical string theory, where summing over the residues at the s-channel poles gives the same answer as summing over the residues at the t-channel poles. We are not aware if this similarity has any further implications.

3. FZZ algebra

Let us now study the linear dilaton theory with a Sine-Liouville perturbation. In previous treatments it has been standard to add to the worldsheet action just one of the two "dressings" of the Sine-Liouville operator.[1] Here we will start by considering simultaneously both sine-Liouville dressings. This point of view has already been demonstrated to be useful in similar contexts (see for example [20]).

As we will see shortly, this approach provides a direct link between Sine-Liouville theory and the 2d black hole. We will demonstrate that the linear dilaton theory perturbed by both dressings of Sine-Liouville operators *requires* the black hole perturbation operator to be turned on for consistency, i.e. exact marginality of the perturbation. So it would seem that the true perturbed theory has both Sine-Liouville and black hole operators turned on at the same time. But this is not the end of the story. A perturbative quantum computation will show that the relative coefficients between all the perturbations are determined self-consistently, in such a way that the coefficient of the second Sine-Liouville perturbation should be set to zero. In other words, the second Sine-Liouville perturbation disappears after fulfilling the role of forcing the black hole perturbation to be present.

We will first describe the general arguments justifying this procedure, and will then show that using various different combinations of the Sine-Liouville operators (of both dressings) and black hole operator as screeners is consistent with the structure of the correlators of the theory and fixes the relative coefficients of the different perturbations. A certain parafermionic symmetry will prove useful in the discussion.

3.1 The algebra of the interactions

The perturbation to the action is as follows[2]

$$S \to S + \int d^2z \left(\mathcal{T}^+_R + \mathcal{T}^+_{-R} + \mathcal{T}^-_R + \mathcal{T}^-_{-R}\right) . \tag{3.1}$$

Between them, these four terms incorporate both signs of the matter momentum as well as both signs of the Liouville dressing. The operators $\mathcal{T}^+_{\pm R}$ were given in eq. (2.16) while the other two are given by:

$$\mathcal{T}^-_{\pm R} = e^{\pm iR(X_L - X_R)}\, e^{(Q+|Q-\frac{1}{Q}|)\phi} , \tag{3.2}$$

As these operators are all of conformal dimension $(1,1)$, the above perturbation is marginal to first order. Now consider the requirements of exact marginality. The general rule is that the perturbations will be exactly marginal if their OPE does not produce another $(1,1)$ operator [22]. However, if they do produce such an operator then we are required to add that operator back into the Lagrangian to restore marginality.

Now it is easily seen that the following OPE holds between the mutually conjugate operators $\mathcal{T}^+_R$ and $\mathcal{T}^-_{-R}$ (a similar relation holds between $\mathcal{T}^-_R$ and $\mathcal{T}^+_{-R}$):

$$\mathcal{T}^+_R(z,\bar{z})\mathcal{T}^-_{-R}(w,\bar{w}) \sim \frac{1}{|z-w|^2}\partial X\bar{\partial}X\, e^{2Q\phi} + \cdots \tag{3.3}$$

[1] In fact, in [6], the dressing is chosen to connect to the semiclassical limit valid for $Q \to \infty$. This is normalizable for $Q < 1$ and becomes non-normalizable for $Q > 1$.

[2] This is really Cosine-Liouville rather than Sine-Liouville.

Here we have exhibited only the $(1,1)$ operator appearing on the r.h.s. . More singular terms correspond to operators that are BRST trivial. Even among the $(1,1)$ operators that can appear, we have dropped BRST-trivial contributions such as $\partial\phi\bar{\partial}\phi\, e^{2Q\phi}$.

On the right hand side of the above equation, we recognise the black hole perturbation operator. This tells us that the Sine-Liouville theory (when viewed as a perturbation of the original action by operators of both Liouville dressings) is not by itself exactly marginal, but marginality can be restored by including the black hole perturbation. In turn, it is known that the latter perturbation can be built up into a solution of closed string field theory which, being unique, must be equivalent to Witten's $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ CFT.[3]

It is worth noting that, as in [23], the operators generated by requiring exact marginality to second order are not quite the physical operators, but rather some variants of them with an extra multiplicative factor of the Liouville field ϕ in front. In the present case the black-hole operator would be replaced by:

$$\partial X\bar{\partial}X e^{2Q\phi} \to \phi\,\partial X\bar{\partial}X e^{2Q\phi}\,. \tag{3.4}$$

This is reminiscent of the fact that, at $c=1$, the cosmological operator in the linear dilaton theory is not really $e^{Q\phi}$ but $\phi\, e^{Q\phi}$. As in that case, the distinction between the operator with and without a ϕ in front is expected to be unimportant for a large class of explicit computations.

3.2 Parafermionic symmetry

To justify and spell out the above observations, we will now perform a more explicit study of the linear dilaton theory perturbed by Sine-Liouville and black hole operators. A useful tool in this study is the fact that when the dilaton slope of ϕ is $Q=\frac{1}{\sqrt{k-2}}$ and the radius of the compact direction is $R=\sqrt{k}$, the symmetry of the worldsheet is expanded from Virasoro to the parafermionic $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ algebra. A representation of this symmetry in terms of the ϕ and X bosons can be obtained by adding one free boson Z, normalized as in (2.9), and starting with the following free-field representation of the level k $\mathrm{SL}(2,\mathbb{R})$ current algebra

$$\begin{aligned} J^3 &= -\sqrt{k}\partial Z\,, \\ J^{\pm} &= (i\sqrt{k}\partial X \mp \sqrt{k-2}\partial\phi)\, e^{\mp\frac{2}{\sqrt{k}}(iX-Z)}\,. \end{aligned} \tag{3.5}$$

Since J^3 corresponds to the direction gauged to obtain the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset, the parafermionic generators can be obtained by dropping Z from the above expressions. This gives

$$\psi^{\pm} = (i\sqrt{k}\partial X \mp \sqrt{k-2}\partial\phi) e^{\mp\frac{2i}{\sqrt{k}}X}\,. \tag{3.6}$$

The currents (3.5) and (3.6) have similar anti-holomorphic copies. A generic primary of the coset can be written in terms of $\mathrm{SL}(2,\mathbb{R})$ quantum numbers as

$$V_{j,m,\bar{m}} = e^{2jQ\phi} e^{-\frac{2im}{\sqrt{k}}X_L} e^{-\frac{2i\bar{m}}{\sqrt{k}}X_R}\,, \tag{3.7}$$

[3] This was demonstrated for $k=9/4$ in [4].

with

$$m = \frac{n+kw}{2} \qquad \bar{m} = \frac{n-kw}{2} \tag{3.8}$$

where n and w are the momentum and the winding of the X direction. This state has conformal dimensions

$$\Delta_{j,m} = -\frac{j(j-1)}{k-2} + \frac{m^2}{k}\,, \tag{3.9}$$

$$\bar{\Delta}_{j,\bar{m}} = -\frac{j(j-1)}{k-2} + \frac{\bar{m}^2}{k}\,, \tag{3.10}$$

and descends to the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset from an $\mathrm{SL}(2,\mathbb{R})$ primary with spin j and $J_0^3 = m, \bar{J}_0^3 = \bar{m}$.

We are interested in turning on marginal perturbations to the flat background which preserve the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ symmetry. Natural candidates are exponentials of X and ϕ. Consider the OPE

$$\begin{aligned}\psi^{\pm}(z)\, e^{iaX_L+b\phi}(w) \sim{}& \frac{\mp b\sqrt{k-2}}{(z-w)^{1\pm\frac{a}{\sqrt{k}}}} e^{b\phi(w)+i(a\mp\frac{2}{\sqrt{k}})X_L(w)}\,(1+O(z-w)) \\ &\mp\frac{k}{2}\partial_z\left(\frac{e^{\mp i\frac{2}{\sqrt{k}}X_L(z)+iaX_L(w)}}{(z-w)^{\pm\frac{a}{\sqrt{k}}}}\right) e^{b\phi(w)}\,. \end{aligned}\tag{3.11}$$

Requiring mutual locality and no single pole fixes $a = w\sqrt{k}$, with w a non-zero integer. Thus the perturbation will be a winding mode belonging to the spectrum of the theory, if we combine left- and right-movers with opposite signs for w. For each w, there are two values of b which give an operator with $\Delta = \bar{\Delta} = 1$. For the case of one unit of winding, the Sine-Liouville operators are

$$S^1_{\pm} \equiv \lambda_1 \mathcal{T}^+_{\pm R} = \lambda_1 e^{\pm i\sqrt{k}(X_L-X_R)+\frac{1}{Q}\phi}\,, \tag{3.12}$$

$$S^2_{\pm} \equiv \lambda_2 \mathcal{T}^-_{\pm R} = \lambda_2 e^{\pm i\sqrt{k}(X_L-X_R)+(2Q-\frac{1}{Q})\phi}\,. \tag{3.13}$$

Turning on Liouville-like perturbations in linear dilaton theories has the effect of screening the strong coupling region $\phi \to +\infty$. This is indeed the case for $S^1_{\pm}$. For $S^2_{\pm}$, this happens only when $2Q > 1/Q$, i.e., $k < 4$. This region includes the $k = 9/4$ value corresponding to the pure two-dimensional black hole. Therefore, we will trust the Lagrangian description of the theory perturbed with $S^2_{\pm}$ in the region $k < 4$, and resort to the analytical continuation of the results otherwise.

The important point is that *both* operators are compatible with the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ symmetry. Now, the chiral black hole perturbation $\partial X e^{2Q\phi}$ should also be compatible with the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ symmetry. This is indeed the case, but happens only at a fixed point of the gauge orbit of the BRST trivial state $\partial\phi e^{2Q\phi}$. To find this point, consider

$$B = (\partial X + \alpha\partial\phi)\, e^{2Q\phi}\,. \tag{3.14}$$

Its OPE with ψ^+ is

$$\psi^+(z)B(w) \sim e^{-2i\frac{X(z)}{\sqrt{k}}+2Q\phi(w)} \left(-\frac{1}{2}\frac{\sqrt{k-2}}{(z-w)^2}+\frac{\partial\phi(w)}{z-w}\right) \times \left(\alpha + i\sqrt{\frac{k-2}{k}}\right), \quad (3.15)$$

and this fixes $\alpha = -i\sqrt{\frac{k-2}{k}}$. For this value of α, the OPE of ψ^- with B is

$$\psi^-(z)B(w) \sim -\frac{i}{\sqrt{k}Q^2}\partial_w \left(\frac{e^{2Q\phi(w)}}{z-w}\right) e^{-2i\frac{X(z)}{\sqrt{k}}} \quad (3.16)$$

so the integrated screening charge $\oint dw B(w)$ also commutes with ψ^-. In the following we rescale B by a constant and add the antichiral factor, so we will use

$$B = \mu \left(i\sqrt{k}\partial X_L + \frac{1}{Q}\partial\phi\right)\left(i\sqrt{k}\bar{\partial} X_R + \frac{1}{Q}\bar{\partial}\phi\right) e^{2Q\phi} \quad (3.17)$$

This, then, is the form of the black hole perturbation that is consistent with the parafermionic symmetry. We will make use of this, along with the Sine-Liouville operators of eqs. (3.12), (3.13), as screening charges in the following subsections.

3.3 Correlation functions

Let us consider the two-point function of the interacting $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ theory with all the perturbations turned on. The vertex operators $V_{j,m,\bar{m}}$ and $V_{-j+1,m,\bar{m}}$ have the same conformal dimension and correspond to incoming and outgoing waves with the same momentum. We normalize them such that

$$\langle V_{-j+1,m,\bar{m}} V_{j,-m,-\bar{m}} \rangle = 1\,, \quad (3.18)$$

and we will consider the two-point function

$$R(j,m,\bar{m}) = \langle V_{j,m,\bar{m}} V_{j,-m,-\bar{m}} \rangle\,. \quad (3.19)$$

We ignore the divergent delta functions in both (3.19) and (3.18). Using the $\mathrm{SL}(2,\mathbb{R})$ quantum numbers is useful because the coset theory inherits the structure of degenerate operators and fusion rules of the $\mathrm{SL}(2,\mathbb{R})$ algebra. This in turn will allow us compute (3.19) by exploiting the trick of Teschner [24, 25, 8].

The affine $\mathrm{SL}(2,\mathbb{R})$ algebra has degenerate primaries at spins [26]

$$j_{r,s} = -\frac{(r-1)}{2} - \frac{(s-1)}{2}k'\,, \quad (3.20)$$

where $k' \equiv k-2$ and r,s are integers with either $r,s>0$ or $r<0, s\leq 0$. The OPE of a primary with spin $j_{r,s}$ gives only a finite number of fields, according to fusion rules that were worked out in [27]. Below we will consider the constraints on the two-point function (3.19) which follow from the degenerate primaries with spins $j=-1/2$ and $j=-k'/2$.

The $j = -1/2$ degenerate field. The fusion of the degenerate primary $V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}$ with any other primary gives [27]

$$V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} V_{j,m,\bar{m}} \sim C^{+}_{j,m,\bar{m}} \left[V_{j-\frac{1}{2},m+\frac{1}{2},\bar{m}+\frac{1}{2}}\right] + C^{-}_{j,m,\bar{m}} \left[V_{j+\frac{1}{2},m+\frac{1}{2},\bar{m}+\frac{1}{2}}\right] . \tag{3.21}$$

Consider the auxiliary three-point function

$$\langle V_{j,m,\bar{m}}(x_1), V_{j+\frac{1}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(x_2) V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(z)\rangle . \tag{3.22}$$

Taking $z \to x_1$ it is equal to

$$C^{-}_{j,m,\bar{m}}\, R\left(j+\frac{1}{2}, m+\frac{1}{2}, \bar{m}+\frac{1}{2}\right) . \tag{3.23}$$

Taking $z \to x_2$ it is equal to

$$C^{+}_{j,m,\bar{m}}\, R(j,m,\bar{m}) . \tag{3.24}$$

Equating the two expressions we get

$$\frac{R(j+\frac{1}{2}, m+\frac{1}{2}, \bar{m}+\frac{1}{2})}{R(j,m,\bar{m})} = \frac{C^{+}_{j,m,\bar{m}}}{C^{-}_{j,m,\bar{m}}} . \tag{3.25}$$

This is a functional equation for $R(j,m,\bar{m})$ that depends on the structure constants $C^{\pm}_{j,m,\bar{m}}$, which, from (3.18) and (3.21), are given by

$$C^{+}_{j,m,\bar{m}} = \langle V_{-j+\frac{3}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(\infty) V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1) V_{j,m,\bar{m}}(0)\rangle , \tag{3.26}$$

$$C^{-}_{j,m,\bar{m}} = \langle V_{-j+\frac{1}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(\infty) V_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1) V_{j,m,\bar{m}}(0)\rangle , \tag{3.27}$$

where

$$V_{j,m,\bar{m}}(\infty) = \lim_{z,\bar{z}\to\infty} z^{2\Delta_{j,m}} \bar{z}^{2\bar{\Delta}_{j,\bar{m}}} V_{j,m,\bar{m}}(z,\bar{z}) \tag{3.28}$$

is the standard BPZ conjugate. In this approach, the computation of $C^{\pm}_{j,m,\bar{m}}$ (and similar constants associated to the second degenerate field below) is the only perturbative result needed, and allows to compare the role of the black hole/Sine-Liouville interactions. The presence of the background charge Q in the ϕ direction implies, for a correlator such as

$$\langle \prod_{i=1}^{n} e^{2\alpha_i Q\phi(z_i)} \rangle , \tag{3.29}$$

the anomalous conservation law

$$\sum_{i=1}^{n} \alpha_i = 1. \tag{3.30}$$

From (3.7) it follows that (3.30) is satisfied for $C^{+}_{j,m,\bar{m}}$ without any insertion of the interactions, so we have $C^{+}_{j,m,\bar{m}} = 1$. For $C^{-}_{j,m,\bar{m}}$, we can satisfy (3.30) by inserting one cigar screening charge (3.17). This gives

$$C^{-}_{j,m,\bar{m}} = \int d^2z \langle V_{-j+\frac{1}{2},-m-\frac{1}{2},-\bar{m}-\frac{1}{2}}(\infty) B(z) V_{\frac{1}{2},\frac{1}{2},\frac{1}{2}}(1) V_{j,m,\bar{m}}(0)\rangle_{\text{free}} \tag{3.31}$$

To compute the integrand, we use the free field correlators

$$
\begin{aligned}
\langle e^{i\frac{2m+1}{\sqrt{k}}X_L(\infty)}e^{-\frac{i}{\sqrt{k}}X_L(1)}e^{-\frac{2im}{\sqrt{k}}X_L(0)}\rangle &= 1\,,\\
i\sqrt{k}\langle e^{i\frac{2m+1}{\sqrt{k}}X_L(\infty)}\partial X_L(z)e^{-\frac{i}{\sqrt{k}}X_L(1)}e^{-\frac{2im}{\sqrt{k}}X_L(0)}\rangle &= -\frac{1/2}{z-1}-\frac{m}{z}\,,\\
\left\langle e^{(-2j+1)Q\phi(\infty)}e^{2Q\phi(z)}e^{-Q\phi(1)}e^{2jQ\phi(0)}\right\rangle &= z^{-2jQ^2}(z-1)^{Q^2}\,,\\
\frac{1}{Q}\left\langle e^{(-2j+1)Q\phi(\infty)}\partial\phi e^{2Q\phi(z)}e^{-Q\phi(1)}e^{2jQ\phi(0)}\right\rangle &= \left(\frac{1/2}{z-1}-\frac{j}{z}\right)z^{-2jQ^2}(z-1)^{Q^2},
\end{aligned}
$$

and similar antiholomorphic expressions. This gives

$$
\begin{aligned}
C^-_{j,m,\bar m} &= \mu(m+j)(\bar m+j)\int d^2z|z|^{-\frac{4j}{k-2}-2}|z-1|^{\frac{2}{k-2}}\\
&= -\mu\frac{\pi}{k'^2}(m+j)(\bar m+j)\gamma\left(-\frac{2j}{k'}\right)\gamma\left(\frac{2j-1}{k'}\right)\gamma\left(1/k'\right)\,, \qquad (3.32)
\end{aligned}
$$

where $\gamma(x)=\Gamma(x)/\Gamma(1-x)$ and we have used (A.4).

We now show that one obtains the same expression for $C^-_{j,m,\bar m}$ using the Sine-Liouville interactions of both dressings as screening charges. The conservation law (3.30) can also be satisfied by inserting one screening of type S^1 and one of type S^2, see (3.12)–(3.13). This gives

$$
\begin{aligned}
C^-_{j,m,\bar m} &= \int d^2zd^2w\langle V_{-j+\frac12,-m-\frac12,-\bar m-\frac12}(\infty)S^1_+(w)S^2_-(z)V_{-\frac12,\frac12,\frac12}(1)V_{j,m,\bar m}(0)\rangle_{\rm free}\\
&= \lambda_1\lambda_2\int d^2zd^2w|z-w|^{-4}|z-1|^{\frac{2}{k-2}}\,z^{m+j-\frac{2j}{k-2}}\,\bar z^{\bar m+j-\frac{2j}{k-2}}\,w^{-m-j}\,\bar w^{-\bar m-j}\ .
\end{aligned}
$$

To compute this integral we can change variables from (z,w) to $(z,y=w/z)$, and we get

$$
\begin{aligned}
C^-_{j,m,\bar m} &= \lambda_1\lambda_2\int d^2y|1-y|^{-4}y^{-m-j}\bar y^{-\bar m-j}\times\int d^2z|z|^{-2-\frac{4j}{k-2}}|1-z|^{\frac{2}{k-2}}\,, \qquad (3.33)\\
&= -\lambda_1\lambda_2\,\Gamma(-1)\frac{\pi^2}{k'^2}(m+j)(\bar m+j)\gamma\left(-\frac{2j}{k'}\right)\gamma\left(\frac{2j-1}{k'}\right)\gamma\left(1/k'\right)\,, \qquad (3.34)
\end{aligned}
$$

where we have used twice eq. (A.4). Thus we have obtained precisely the same expression (3.32) for $C^-_{j,m,\bar m}$ using both Sine-Liouville screenings, and we can identify

$$\mu=\lambda_1\tilde\lambda_2\,, \qquad (3.35)$$

where

$$\tilde\lambda_2=\pi\Gamma(-1)\lambda_2 \qquad (3.36)$$

is a renormalized value of λ_2. The reason to renormalize only λ_2 in the product $\lambda_1\lambda_2$ will become clear below.

JHEP01(2007)025

The $j=-k'/2$ degenerate field. For this case, the fusion rules are [27]

$$V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}}V_{j,m,\bar{m}} \sim \tilde{C}^{+}_{j,m,\bar{m}}\left[V_{j-\frac{k'}{2},m+\frac{k'}{2},\bar{m}+\frac{k'}{2}}\right]+\tilde{C}^{-}_{j,m,\bar{m}}\left[V_{j+\frac{k'}{2},m+\frac{k'}{2},\bar{m}+\frac{k'}{2}}\right]$$
$$+\tilde{C}^{\times}_{j,m,\bar{m}}\left[V_{\frac{k'}{2}-j+1,m+\frac{k'}{2},\bar{m}+\frac{k'}{2}}\right]\ . \tag{3.37}$$

A similar reasoning as that used above leads leads to the functional equation

$$\frac{R(j+\frac{k'}{2},m+\frac{k'}{2},\bar{m}+\frac{k'}{2})}{R(j,m,\bar{m})}=\frac{\tilde{C}^{+}_{j,m,\bar{m}}}{\tilde{C}^{-}_{j,m,\bar{m}}}\,. \tag{3.38}$$

As before, we can set

$$\tilde{C}^{+}_{j,m,\bar{m}}=\langle V_{-j+\frac{k'}{2}+1,-m-\frac{k'}{2},-\bar{m}-\frac{k'}{2}}(\infty)V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}}(1)V_{j,m,\bar{m}}(0)\rangle=1\,, \tag{3.39}$$

since the conservation law (3.30) is satisfied without any perturbative insertion. As for

$$\tilde{C}^{-}_{j,m,\bar{m}}=\langle V_{-j-\frac{k'}{2}+1,-m-\frac{k'}{2},-\bar{m}-\frac{k'}{2}}(\infty)V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}}(1)V_{j,m,\bar{m}}(0)\rangle\,, \tag{3.40}$$

we can satisfy (3.30) by inserting two Sine-Liouville S^1 interactions. This gives

$$\tilde{C}^{-}_{j,m,\bar{m}}=\int d^2zd^2w\langle V_{-j-\frac{k'}{2}+1,-m-\frac{k'}{2},-\bar{m}-\frac{k'}{2}}(\infty)S^1_{-}(w)S^1_{+}(z)V_{-\frac{k'}{2},\frac{k'}{2},\frac{k'}{2}}(1)V_{j,m,\bar{m}}(0)\rangle_{\text{free}}$$
$$=\lambda_1^2\int d^2zd^2w|z-w|^{-2k+2}|z-1|^{2k'}\,z^{m-j}\,\bar{z}^{\bar{m}-j}\,w^{-m-j}\,\bar{w}^{-\bar{m}-j}\ . \tag{3.41}$$

Changing now variables from (z,w) to $(z,y=w/z)$ we get

$$\tilde{C}^{-}_{j,m,\bar{m}}=\lambda_1^2\int d^2z|z|^{-4j-2k'}|z-1|^{2k'}\times\int d^2y^{-m-j}\bar{y}^{-\bar{m}-j}|1-y|^{-2(k-1)}\ ,$$
$$=\lambda_1^2\pi^2\gamma\left(-2j-k'+1\right)\gamma\left(2j-1\right)\frac{\Gamma(-m-j+1)}{\Gamma(-m-j-k'+1)}\frac{\Gamma(\bar{m}+j+k')}{\Gamma(\bar{m}+j)}\ , \tag{3.42}$$

where we have used (A.4) twice.

Now that we have the structure constants, the solution to the functional equations (3.25) and (3.38) is

$$R(j,m,\bar{m})=(\mu\pi\gamma(1/k'))^{1-2j}\frac{\Gamma(2j-1)\Gamma(1+\frac{2j-1}{k-2})}{\Gamma(-2j+1)\Gamma(1-\frac{2j-1}{k-2})}\frac{\Gamma(-j+1+\bar{m})\Gamma(-j+1-m)}{\Gamma(j+\bar{m})\Gamma(j-m)} \tag{3.43}$$

which is the expression we wrote above in (2.17). Also, we get also the relation

$$\lambda_1^2\pi^2=(\mu\pi\gamma(1/k'))^{k'} \tag{3.44}$$

which was first obtained in [8] by similar methods. Given this relation, it is clear that λ_2 rather than λ_1 is the coefficient which should absorb the divergence coming from $\Gamma(-1)$ in (3.34). But since from (3.35) we see that $\tilde{\lambda}_2$ is finite, it follows from (3.36) that λ_2 is effectively renormalized to zero, and therefore the second Sine-Liouville screening disappears from the theory.

JHEP01(2007)025

Also note from (3.35) and (3.44) that only one of the three coefficients $\mu, \lambda_1, \tilde{\lambda}_2$ is independent. The expression (3.43) for $R(j,m,\bar{m})$ is symmetric under $m \leftrightarrow \bar{m}$ for $m-\bar{m} \in \mathbb{Z}$, using (A.1). It satisfies $R(-j+1,m,\bar{m}) = R^{-1}(j,m,\bar{m})$, and for delta normalizable states ($j = \frac{1}{2} + i\mathbb{R}$) it is a phase, namely, the phase shift between an incoming and an outgoing wave.

Using the Teschner trick one can also obtain the three-point function, and the same special structure constants we computed above enter similarly as an input for functional relations for the three-point function, which follow from crossing symmetry of an auxiliary four-point function [28].

Therefore, in this efficient approach to compute the correlators, the role of the second Sine-Liouville dressing is established for two and three point functions. It would be interesting to use the second Sine-Liouville screening to perform free-field computations similar to those in [7].

4. Generalized FZZ algebra

As an application of the considerations detailed above, we will investigate a generalized class of Sine-Liouville models. The FZZ algebra procedure will then be employed to find the analogs of the dual black hole operators. However, such operators exist only in the special case of $c = 1$ matter coupled to Liouville theory. Therefore we will first give a brief survey of the relevant aspects of $c = 1$ string theory, including a listing of some interesting physical states in the cohomology, before proceeding to the model.

4.1 Cohomology of $c = 1$ strings

The $c = 1$ string is a special case of the linear dilaton background of Section 2.2 where we set $Q = 2$ to get a total central charge $c = 26$. With a cosmological perturbation to cut off the strong coupling region, the worldsheet action is:

$$S_{\mathrm{c}=1} = \int d^2z \left(-\partial X \bar{\partial} X + \partial\phi\bar{\partial}\phi + 2\hat{R}(z,\bar{z})\phi + 4\pi\mu\, e^{2\phi} \right) . \tag{4.1}$$

The string loop expansion in this theory is an expansion in $\frac{1}{\mu^2}$. The coordinate X has the interpretation of time, but in what follows we will consider its Euclidean continuation, corresponding to the case of finite temperature. Thus X is Euclidean (spacelike) and compactified:

$$X(z,\bar{z}) \sim X(z,\bar{z}) + 2\pi R\,. \tag{4.2}$$

The physical fields of the theory are defined by the BRST procedure, which is most tractable when the worldsheet theory is a free field theory. In the present case, the theory (at least in the given variables) is free only at $\mu = 0$, the limit in which the effective string coupling is infinite. For this case, the BRST cohomology has been worked out in [29 – 34]. As observed in [35], at nonzero μ we can still use part of the previous results.

JHEP01(2007)025

Let us therefore start by reviewing the cohomology at $\mu = 0$. One important class of physical operators[4] are the momentum "tachyons":

$$T^{\pm}_{\frac{n}{R}} = e^{i\frac{n}{R}X}\, e^{(2\mp\frac{n}{R})\phi}, \quad n \in \mathbb{Z} \tag{4.3}$$

with left and right conformal dimensions equal to 1. These are just the special cases of the operators already introduced in eqs. (2.16), (3.2). As before, the superscripts $\pm$ refer to non-normalizable/normalizable operators respectively.

Another important class of observables are the winding modes. Writing $X = X_L + X_R$, we define $\tilde{X} = X_L - X_R$ in terms of which:

$$\mathcal{T}^{\pm}_{nR} = e^{i\,nR\,\tilde{X}}\, e^{(2\mp nR)\,\phi}, \quad n \in \mathbb{Z}\,. \tag{4.4}$$

These are clearly also $(1,1)$ operators.

The operators $T_{\frac{n}{R}}$ and $\mathcal{T}_{nR}$ are dual to each other under (timelike) T-duality:

$$X_R \to -X_R, \quad \phi \to \phi - \log R \tag{4.5}$$

under which $X \to \tilde{X}$ and

$$R \to \frac{1}{R}, \quad \mu \to \mu R\,. \tag{4.6}$$

Note that $T_0 = \mathcal{T}_0 = e^{2\phi}$ is the cosmological operator.

There are other modes of dimension $(1,1)$. They are called "discrete states" [32, 33] and can be thought of as two-dimensional "remnants" of the higher-spin fields that exist in critical string theory. We start by writing the following chiral operators at the self-dual radius $R = 1$

$$W^{\pm}_{s,n}(z) = \mathcal{P}_{s,n}(\partial^j X)\, e^{2inX_L}\, e^{(2\mp 2s)\phi_L} \tag{4.7}$$

where $s = 0, \frac{1}{2}, 1, \ldots$, and $n, n' = s, s-1, \ldots, 1-s, -s$ and $\mathcal{P}_{s,n}$ is a polynomial in derivatives of X_L with conformal dimension $s^2 - n^2$. In particular, $\mathcal{P}_{s,\pm s} = 1$.

Because the above operators depend only on the left-moving part of the Liouville field, which is a noncompact scalar field, they are not physical operators. The physical operators are the combinations:

$$Y^{\pm}_{s;n,n'}(z,\bar{z}) = W^{\pm}_{s,n}(z)\bar{W}^{\pm}_{s,n'}(\bar{z})\,. \tag{4.8}$$

For $n = n' = \pm s$ the above operators are the momentum modes $T_{\pm 2s}$, while for $n = -n' = \pm s$ they are the winding modes $\mathcal{T}_{\pm 2s}$. The remaining ones, with $n < s$ or $n' < s$ are the true discrete states. The time-independent discrete states are those with $n = n' = 0$. Simple examples are the ones with $s = 1, 2$ for which the relevant $c = 1$ primaries are:

$$\begin{aligned} \mathcal{P}_{1,0} &= \partial X\,, \\ \mathcal{P}_{2,0} &= (\partial X)^4 + \tfrac{3}{2}(\partial^2 X)^2 - 2\partial X \partial^3 X\,, \end{aligned} \tag{4.9}$$

[4] Here and in what follows, we refer to a dimension $(1,1)$ operator as a physical operator if it is BRST invariant after integration over the worldsheet. Typically such operators are also BRST invariant when multiplied by the ghost field combination $c\bar{c}$. We need to be more specific about the ghost dependence of a physical operator only if this dependence is nontrivial.

and the corresponding non-normalizable discrete-state operators are:

$$Y^{+}_{1;0,0} = \partial X\bar{\partial}X\,, \tag{4.10}$$

$$Y^{+}_{2;0,0} = \mathcal{P}_{2,0}\bar{\mathcal{P}}_{2,0}\,e^{-2\phi}\,. \tag{4.11}$$

Although the above states have been tabulated for a specific radius $R = 1$, they will exist at other radii as long as $n+n'$ is an integer multiple of $1/R$ and $n-n'$ is an integer multiple of R. In particular this constraint is always satisfied for $n = n' = 0$, independent of the radius, hence the time-independent discrete states $Y^{+}_{s;0,0}$ exist for all radius. Of course s has to be an integer in order for $n = n' = 0$ to be allowed. Since we are working at general values of the radius, we will concentrate on this set of time-independent discrete states.

The first nontrivial state in this collection is just $Y^{+}_{1,0,0} = \partial X\bar{\partial}X$, the radius-changing operator. In the critical string this would have just been the zero-momentum mode of the graviton/dilaton. Here it is a “remnant” of those fields, and is forced to have zero momentum. The other discrete states are similar remnants of excited tensor states of the string, with fixed momenta.

Note that for the radius operator appearing in eq. (4.10) there is a normalizable, or non-local, counterpart:

$$Y^{-}_{1,0,0} = \partial X\bar{\partial}X\,e^{4\phi}\,. \tag{4.12}$$

This is precisely the black hole perturbation of the previous sections, specialised to the case $Q = 2$. It has been shown [4] that starting from a perturbation of the $c = 1$ string by $Y^{-}_{1,0,0}$, there is no obstruction to finding a classical solution of closed string field theory (CSFT) to all orders in α', and moreover the solution so obtained is unique. Therefore, starting with eq. (2.6) one generates an exact (at tree level) CFT describing a string background. It follows that this CFT must be the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ black hole CFT. This closes the gap between the spacetime solution, valid only for large k, and the CFT, which lacks a direct spacetime interpretation.

But it also suggests a generalization. Observe that:

$$Y^{-}_{s;0,0} = \mathcal{P}_{s,0}\bar{\mathcal{P}}_{s,0}(\partial^{j}X,\bar{\partial}^{j}X)\,e^{(2+2s)\phi} \tag{4.13}$$

for $s = 0, 1, 2, \ldots$ defines an infinite family of normalizable operators, of which the first two ($s = 0, 1$) are the cosmological and black hole perturbations. Now the considerations in ref. [4] were shown to be generally applicable to all these operators. Therefore each of them similarly generates a unique classical solution of CSFT, and so must correspond to some exact CFT. Unlike the first nontrivial case ($s = 1$, the usual 2d black hole) where the CFT is the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ nonlinear σ-model, the CFT in the other cases is not explicitly known. The form of the states in eq. (4.13) suggests that we are dealing with higher-spin generalizations of the 2d black hole. As we will now argue, these are related by a generalized FZZ duality to Sine-Liouville perturbations of higher winding number.

4.2 Higher winding Sine-Liouville perturbations

Supposing that instead of the unit winding perturbation $V = \mathcal{T}_1$, we perturb the action of the linear dilaton theory by Sine-Liouville operators of winding number 2:

$$\mathcal{T}^{\pm}_{\pm 2R} = e^{\pm 2iR(X_L - X_R)}\,e^{(2\mp 2R)\phi} \tag{4.14}$$

(recall that the $\pm$ sign in the subscript refers to the sign of the winding number while the one in the superscript refers to the dressing). It is easily checked that the OPE between mutually conjugate operators of this type is again the black hole perturbation:

$$\mathcal{T}^{+}_{2R}(z,\bar{z})\mathcal{T}^{-}_{-2R}(w,\bar{w}) \sim \frac{1}{|z-w|^2}\partial X\bar{\partial}X\, e^{4\phi} + \cdots \tag{4.15}$$

The same will be true for pairs of mutually conjugate operators of any winding number — in every case, the output of the OPE is the 2d black hole perturbation. One way to understand this is that we can orbifold the compact time direction to enhance the radius by an integer factor. The multiply wound Sine-Liouville perturbation of the original theory then become singly-wound perturbations in the orbifolded theory. But orbifolding in time does not affect the black hole perturbation operator, which is time-independent.

Things become more interesting if we perturb the theory simultaneously by operators of different winding numbers. As a first example, consider the theory perturbed by the single and double-winding operators:

$$S \to S + \int d^2z\,(\mathcal{T}^{+}_{R} + \mathcal{T}^{+}_{-R} + \mathcal{T}^{-}_{R} + \mathcal{T}^{-}_{-R} + \mathcal{T}^{+}_{2R} + \mathcal{T}^{+}_{-2R} + \mathcal{T}^{-}_{2R} + \mathcal{T}^{-}_{-2R}) \tag{4.16}$$

In this case, examining the OPE algebra, we find that the product of *three* of these operators can potentially produce a new $(1,1)$ operator on the r.h.s. :

$$\mathcal{T}^{\pm}_{-2R}(z_1,\bar{z}_1)\mathcal{T}^{\mp}_{R}(z_2,\bar{z}_2)\mathcal{T}^{\mp}_{R}(z_3,\bar{z}_3) \;\sim\; \mathcal{P}_{2,0}(\partial X)\bar{\mathcal{P}}_{2,0}(\bar{\partial}X)\, e^{6\phi} = Y^{-}_{2,0,0} \tag{4.17}$$

where $\mathcal{T}^{\pm}_{nR}$ can be read off from eq. (4.4) and $\mathcal{P}_{2,0}$ is given explicitly in eq. (4.9).

Let us work this out in more detail. We have:

$$\begin{aligned}\mathcal{T}^{+}_{2R}(z_1,\bar{z}_1)\mathcal{T}^{-}_{-R}(z_2,\bar{z}_2)\mathcal{T}^{-}_{-R}(z_3,\bar{z}_3) &=: e^{2iRX_1}e^{(2-2R)\phi_1} :: e^{-iRX_2}e^{(2+R)\phi_2} :: e^{-iRX_3}e^{(2+R)\phi_3} : \\ &= \frac{1}{|z_{12}|^{4-2R}}\frac{1}{|z_{13}|^{4-2R}}\frac{1}{|z_{23}|^{4+4R}} : e^{iR(2X_1-X_2-X_3)} :: e^{(2-2R)\phi_1+(2+R)\phi_2+(2+R)\phi_3} :\end{aligned} \tag{4.18}$$

where we have used the shorthand $X_i \equiv X(z_i,\bar{z}_i)$ and similarly for ϕ_i, as well as $z_{ij} \equiv z_i - z_j$.

After integration over the z_i, the r.h.s. of the above may be written

$$\int \prod_{i=1}^{3} d^2w_i \frac{1}{|w_1|^{4-2R}}\frac{1}{|w_2|^{4-2R}}\frac{1}{|w_1-w_2|^{4+4R}}\mathcal{O}(w_i,\bar{w}_i) \tag{4.19}$$

where we have defined $w_1 \equiv z_1 - z_2, w_2 \equiv z_1 - z_3, w_3 = z_3$, and

$$\mathcal{O}(w_i,\bar{w}_i) \equiv \Big[: e^{iR(2X(w_2+w_3)-X(w_2-w_1+w_3)-X(w_3))} : \times\, (w \to \bar{w})\Big] : e^{6\phi(w_3,\bar{w}_3)} : \tag{4.20}$$

Here we have moved all the Liouville fields ϕ_i to the location w_3 and dropped the new terms that arise in doing this. As we will see, at the end this will only lose us some terms that are trivial in the BRS cohomology.

Finally we expand the X-dependent vertex operator about the point w_3 as:

$$\Big[: e^{iR(2X(w_2+w_3)-X(w_2-w_1+w_3)-X(w_3))} : \times\, (w \to \bar{w})\Big] := \Big| \sum_{n_1,n_2=0}^{\infty} w_2^{n_1}(w_2-w_1)^{n_2} \mathcal{A}_{n_1,n_2}(w_3)\Big|^2 \tag{4.21}$$

where the operators $\mathcal{A}_{n_1,n_2}$ are built out of holomorphic derivatives of X, namely $\partial X, \partial^2 X, \dots$ and have conformal dimension $(\Delta, \bar{\Delta}) = (n_1 + n_2, 0)$. Their complex conjugates have dimension $(0, n_1+n_2)$. Anticipating that the final contribution can only come from physical operators in the cohomology, we keep only those operators in the sum which are of the form $|\mathcal{A}_{n_1,n_2}|^2$ with $n_1 + n_2 = 4$. Then the combined matter-Liouville operator in eq. (4.20) can be replaced by:

$$|w_2|^{2n_1}|w_1 - w_2|^{2n_2}|\mathcal{A}_{n_1,n_2}(w_3)|^2 \ : e^{6\phi(w_3,\bar{w}_3)} : \tag{4.22}$$

Now we see that the composite operator

$$|\mathcal{A}_{n_1,n_2}(w_3)|^2 \ : e^{6\phi(w_3,\bar{w}_3)} : \tag{4.23}$$

has conformal dimension $(1, 1)$ and is a local operator depending only on $(w_3, \bar{w}_3)$. The coefficient functions depend only on w_1, w_2 and combine under the integral sign into an expression of the form:

$$\int d^2w_1 d^2w_2 \frac{1}{|w_1|^{\alpha}} \frac{1}{|w_2|^{\beta}} \frac{1}{|w_1 - w_2|^{\gamma}} \tag{4.24}$$

for some α, β, γ satisfying $\alpha + \beta + \gamma = 4$. Thus the coefficient of the $(1, 1)$ operator is logarithmically divergent, the sign of a nontrivial β-function.

At this stage it is clear without further computation that the operator $\mathcal{A}_{n_1,n_2}$ must be the Virasoro primary $\mathcal{P}_{2,0}$ defined in eq. (4.9). The reason is that the three operators whose OPE we are computing are all in the cohomology and the output must therefore also be in the cohomology. Given the total matter and Liouville momenta of the fields on the l.h.s. of the multiple OPE, there is a unique such operator that can appear on the r.h.s. . Hence we have shown that the higher-spin black hole operator

$$Y^-_{2,0,0} = \mathcal{P}_{2,0}(\partial X)\bar{\mathcal{P}}_{2,0}(\bar{\partial} X)\, e^{6\phi} \tag{4.25}$$

appears in the β-function of the theory perturbed as in eq. (4.16), thereby justifying eq. (4.17).

The above result is quite general. For example, one can check that:

$$\mathcal{T}^{\pm}_{NR}(z_1, \bar{z}_1)\mathcal{T}^{\mp}_{-R}(z_2, \bar{z}_2)\cdots \mathcal{T}^{\mp}_{-R}(z_N, \bar{z}_N) \ \sim \ \mathcal{P}_{N,0}(\partial X)\bar{\mathcal{P}}_{N,0}(\bar{\partial} X)\, e^{(2+2N)\phi} = Y^-_{N,0,0} \tag{4.26}$$

Thus the higher-spin black hole operator of label N (i.e. Liouville momentum $2{+}2N$) arises when we perturb the linear dilaton theory with Sine-Liouville operators of windings 1 and N.

We see that, in a similar sense as for the FZZ algebra of the previous section, the multiply-wound Sine-Liouville operators are linked to higher-spin black holes. More precisely, perturbing by all Sine-Liouville operators of winding numbers $1, 2, \dots, N$ gives rise to higher-spin black holes with all labels up to N (the spins realised in this way are $2k^2,\ k = 1, 2 \cdots, N$). This should be viewed as a generalization of the FZZ duality, and we expect that also here the coefficients of half of the Sine-Liouville operators get renormalized to zero.

To produce only a definite higher-spin black hole for $N \geq 2$, one must fine-tune the perturbation strengths so that the lower-spin operators are not produced.

5. Conclusions

In this work we presented a new approach to the FZZ duality between the two-dimensional black hole and the sine-Liouville conformal field theory. In this approach the duality is to be understood as coming from the fact that the Sine-Liouville perturbations of both dressings induce, via their mutual OPE, the operator representing a black hole deformation, and one the of the two Sine-Liouville perturbations then disappears because its coefficient gets renormalized to zero.

This approach has led us to propose a generalized FZZ duality for the $c = 1$ string. One side of this duality is a CFT generated by perturbing the linear dilaton background with higher-spin analogues of the black hole operator. Finding an exact description of this CFT would be very helpful in understanding this generalized duality better, though that appears to be a hard problem on which no progress has been made since the existence proof in ref. [4]. One might instead try to use the holographic description in terms of double-scaled matrix models to get more insight into the nature of the theory that results from the fully back-reacted higher spin perturbation. Also, even partial progress in the worldsheet treatment of the higher spin perturbations (e.g. some correlation functions) could provide relevant tests for the generalized FZZ duality we have proposed.

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A. Useful formulae

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \tag{A.1}$$

$$\gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1-x)} \tag{A.2}$$

$$\gamma(x+1) = -x^2\gamma(x) \tag{A.3}$$

$$\begin{aligned}\int_{\mathbb{R}^2} d^2x\, x^a \bar{x}^{\bar{a}} (1-x)^b (1-\bar{x})^{\bar{b}} &= \pi \frac{\Gamma(1+a)}{\Gamma(-\bar{a})} \frac{\Gamma(1+b)}{\Gamma(-\bar{b})} \frac{\Gamma(-\bar{a}-\bar{b}-1)}{\Gamma(a+b+2)} \\ &= (a, b \longleftrightarrow \bar{a}, \bar{b})\end{aligned} \tag{A.4}$$

The above integral is well defined only when $a - \bar{a}, b - \bar{b} \in Z$, and it is only then that the second line holds.

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JHEP01(2007)025